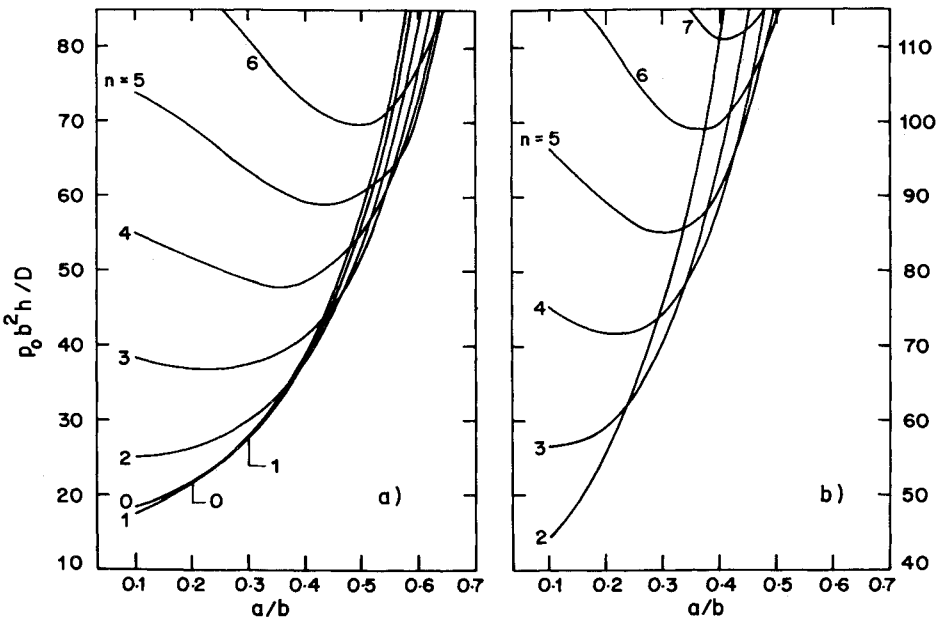

Fig. 4 Variation of buckling load parameter with hole size: a) both edges simply supported (SS); b) both edges clamped (CC).

critical buckling loads of plates supported along appropriate number of regularly spaced diametral lines.

Out of the remaining six cases (Figs. 2–4), the plate buckles axisymmetrically in the CF case for hole sizes up to about 0.5 (Fig. 2a) and in the SS case for hole sizes between 0.2–0.3 (Fig. 4a). For other hole sizes in the CF and SS cases and for all hole sizes in the remaining four cases, the plate buckles with one or more circumferential waves. In such cases, the first axisymmetric eigenmode and the corresponding eigenload have no physical significance whatsoever. However, it may be mentioned here that the higher axisymmetric modes do correspond to the critical buckling modes of plates supported along the relevant nodal circles. The modes with higher number of circumferential waves considered in the present investigation have similar physical significance, as now explained. Let $(a/b)_n$ denote the hole size of the plate for which the two modes with n and $2n$ number of circumferential waves yield equal buckling loads. Then for a/b less than $(a/b)_n$, the mode with n circumferential waves is the critical buckling mode of the plate supported along n regularly spaced diametral lines. For a/b greater than $(a/b)_n$, this mode and the corresponding eigenvalue have no physical significance.

From Figs. 2–4 it can also be seen that the number of circumferential waves of the buckled surface increases, in general, with increasing hole size and with increasing geometric constraints at the edges. For these critical buckling modes of plates with large holes, Majumdar[4] proposed one term Rayleigh-Ritz procedure in the CF case and stated that such a procedure "provides a reasonable estimate of the minimum buckling loads for at least up to values of $a/b = 0.9$ and $n = 10$." However, such estimates have been found to be more than 5 percent higher than the present accurate estimates.

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Numerical Methods for Evaluating the Derivatives of Eigenvalues and Eigenvectors

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I. Introduction

MATHEMATICAL programing methods for optimizing the design of systems where the dynamic stability and/or response of the system is a function of several design parameters often require derivatives of the eigenvalues and eigenvectors of a characteristic equation of the system. Cwach and Stearman[1] utilized the first derivatives of the eigenvalues of the flutter equation of a lifting surface to find gradients of the damping factor of the flutter characteristic equation. These gradients were used to optimize an active control system which suppressed the flutter of a lifting surface. Rudisill and Bhatia[2,3] utilized

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the first and second derivatives of the eigenvalues of the governing equations for steady-state oscillations of an aircraft structure to find the first and second derivatives of the flutter velocity. These derivatives of the flutter velocity were used in gradient search procedures to search for a minimum mass structure which would have a specified flutter velocity. Rudisill and Cooper[4] developed a procedure for determining the flutter velocity of an aircraft structure which utilized the first and second derivatives of the eigenvalues of the flutter equation.

This Note is concerned with the presentation of two numerical methods for computing the derivatives of eigenvalues and eigenvectors. The first method to be presented is an iteration method for finding the first partial derivatives of the eigenvalues and eigenvectors of a self-adjoint system. The iteration method will also find the first partial derivative of the largest eigenvalue and its corresponding eigenvector of a nonself-adjoint system, but the method cannot be used to find the derivatives of the remaining $n-1$ eigenvalues and eigenvectors. The first derivative of the eigenvalues and eigenvectors may be then used to find the second derivatives of their corresponding eigenvalues. The method requires only the first m eigenvectors of a system if the derivatives of the first m eigenvectors are sought. The second method to be presented is very similar to a method recently presented by Rudisill[5] for finding the derivatives of the eigenvalues and eigenvectors of *nonself-adjoint* systems. The second method requires only one eigenvector to find all orders of its derivatives and the derivatives of its corresponding eigenvalue. The method requires the solution of $n+1$ simultaneous equations where n is the number of coordinates of an eigenvector.

II. The Iterative Method

Consider the equation

$$(A-\lambda_i l)X_i = 0 \tag{1}$$

where A is an $n \times n$ matrix which is a function of the parameter $P_j (j = 1,2,\ldots,m)$, l is the identity matrix, λ_i is an eigenvalue of the characteristic determinant,

$$|A-\lambda_i l| = 0 \tag{2}$$

and X_i is the corresponding right-hand eigenvector of λ_i. The left-hand eigenvector of the matrix A is defined by the relation

$$y_i^t(A-\lambda_i l) = 0 \tag{3}$$

where the superscript t indicates the transpose of the vector y_i. If the left-hand eigenvector y_i satisfies Eq. (1) when it is substituted in place of X_i then the system of eigenvector will be defined here as self-adjoint. The eigenvectors of a self-adjoint system are orthogonal.

The eigenvalues of Eq. (1) are assumed to be numbered in a descending order of their absolute values and the eigenvectors are normalized such that

$$X_i^t X_i = 1 \tag{4}$$

Differentiating Eq. (1) with respect to the parameter P_j and premultiplying the results by the transpose of X_i yields the relation

$$\lambda_{i,j} = X_i^t A_{,j} X_i + X_i^t(A-\lambda l)X_{i,j} \tag{5}$$

where the subscript $,j$ indicates the partial derivative with respect to P_j. If the system is self-adjoint then the second term of the right-hand side of Eq. (5) is zero and $\lambda_{i,j}$ is determined if X_i is known.

The following relation may be formed by differentiating Eq. (1) with respect to P_j

$$X_{i,j} = [(A_{,j}-\lambda_{i,j} l)X_i + AX_{i,j}]/\lambda_i \tag{6}$$

From Eqs. (5) and (6) the following two recursion equations may be formed

$$\lambda_{i,j}^k = X_i^t A_{,j} X_i + X_i^t(A-\lambda_i l)X_{i,j}^k \tag{7}$$

$$X_{i,j}^{k+1} = \{(A_{,j}-\lambda_{i,j}^k l)X_i + AX_{i,j}^k\}/\lambda_i \tag{8}$$

where $X_{i,j}^k$ is the kth trial value of $X_{i,j}$.

It is our experience that for an arbitrarily assumed initial value of $X_{i,j}$ the recursion Eqs. (7) and (8) (when substituted into an iterative cyclic fashion) will cause $\lambda_{i,j}^k$ and $X_{i,j}^{k+1}$ to converge to $\lambda_{i,j}$ and $X_{i,j}$ when λ_i is the first (largest) eigenvalue. Obviously when the system is self-adjoint $\lambda_{i,j}^k$ would be computed only one time from the relation

$$\lambda_{i,j}^k = X_i^t A_{,j} X_i \tag{9}$$

Derivatives of the eigenvalues and eigenvectors other than the first eigenvalue and eigenvector will not generally converge when only Eqs. (7) and (8) are employed in an iterative cycle and if convergence does occur the derivatives of the eigenvectors are not necessarily unique. Additional conditions must be imposed to force the convergence of the derivatives of the remaining $n-1$ eigenvectors.

An iterative procedure for finding the derivatives of the remaining $n-1$ eigenvectors of a *self-adjoint* system will be presented next. Define a new set of vectors $\bar{X}_i$ such that

$$\begin{aligned} \bar{X}_1 &= X_1 \\ \bar{X}_2 &= X_2 + a_{21}\bar{X}_1 \\ &\vdots \\ \bar{X}_n &= X_n + a_{n1}\bar{X}_1 + \ldots + a_{nn-1}\bar{X}_{n-1} \end{aligned} \tag{10}$$

Differentiating Eqs. (10) with respect to P_j yields the relations

$$\begin{aligned} \bar{X}_{1,j} &= X_{1,j} \\ \bar{X}_{2,j} &= X_{2,j} + a_{21,j}\bar{X}_1 + a_{21}\bar{X}_{1,j} \\ &\vdots \\ \bar{X}_{n,j} &= X_{n,j} + a_{n1,j}\bar{X}_1 + \ldots + a_{nn-1}\bar{X}_{n-1,j} \end{aligned} \tag{11}$$

Since the eigenvectors of a self-adjoint system are orthogonal

$$X_i^t X_k = 0 \qquad i \neq k \tag{12}$$

then

$$X_{i,j}^t X_k + X_i^t X_{k,j} = 0 \tag{13}$$

If Eq. (4) is differentiated with respect to P_j then

$$X_{i,j}^t X_i = 0 \tag{14}$$

If it is now required that the system of vectors $\bar{X}_i$ be orthogonal then it is easily shown by use of Eqs. (13) and (14) that all values of a_{kh} and their derivatives are zero and $\bar{X}_i = X_i$; however if the system of eigenvectors $\bar{X}_i$ are required to be orthogonal, then a set of recursion equations may be formed from Eqs. (11) by setting the coefficients a_{kh} equal to zero; i.e.,

$$\begin{aligned} X_{2,j}^{k+1} &= X_{2,j}^k + a_{21,j}X_1 \\ X_{3,j}^{k+1} &= X_{3,j}^k + a_{31,j}X_1 + a_{32,j}X_2 \\ &\vdots \\ X_{n,j}^{k+1} &= X_{n,j}^k + a_{n1,j}X_1 + \ldots + a_{nn-1,j}X_{n-1,j} \end{aligned} \tag{15}$$

The coefficient $a_{21,j}$ may be evaluated by first premultiplying the first equation of Eq. (15) by X_1^t and then eliminating $X_i^t X_{2,j}^{k+1}$ from the resulting expression by use of Eq. (13). Then it is seen that

$$a_{21,j} = -X_{1,j}^t X_2 - X_1^t X_{2,j}^k \tag{16}$$

From Eqs. (13) and (16) it is seen that $a_{21,j}$ will approach zero as $X_{2,j}^k$ converges to the true value.

The derivative of the eigenvalue λ_2 may be computed from Eq. (9). Then Eqs. (16), the first equation of Eqs. (15) and Eq. (8) may be used in an iterative cycle in the order listed to evaluate $X_{2,j}$.

The coefficients $a_{31,j}$ may be evaluated by first premultiplying the second equation of Eqs. (15) by X_1^t and then eliminating $X_1^t X_{3,j}^{k+1}$ from the resulting expression, then

$$a_{31,j} = -X_{1,j}^t X_3 - X_1^t X_{3,j}^k \tag{17}$$

The coefficients $a_{32,j}$ may be found in a similar manner by first premultiplying the second of Eqs. (15) by X_2^t and then eliminating $X_2^t X_{3,j}^{k+1}$ by use of Eq. (13), then

$$a_{32,j} = -X_{2,j}^t X_3 - X_2^t X_{3,j}^k \tag{18}$$

This procedure may be used to show that

$$a_{ih,j} = -X_{h,j}^t X_i - X_h^t X_{i,j}^k \tag{19}$$

where $i \neq h$. The procedure used for finding the derivative of X_2 may be continued to find the derivatives of all the eigenvectors X_i.

It is obvious that the preceding procedure would be very lengthy if the derivatives of all of the eigenvectors are sought when n is large; however, if only the derivatives of the first few eigenvectors are sought then the process may prove to be economical.

The second derivative of the eigenvalues may be expressed in the form‡

$$\lambda_{i,jk} = [y_i^t A_{,jk} X_i + y_i^t (A_{,j} - \lambda_{i,j} I) X_{i,k} + y_i^t (A_{,k} - \lambda_{i,k} I) X_{i,j}] / y_i^t X_i \quad (20)$$

When the system is self-adjoint X_i^t may be substituted for y_i^t in Eq. (20).

III. Algebraic Method

Consider the matrix equation

$$(A - \lambda_i B) X_i = 0 \quad (21)$$

where A and B are $n \times n$ matrices such that all of the eigenvalues λ_i are distinct, and the system is not necessarily self-adjoint. The first partial derivative of Eq. (21) with respect to P_j may be written in the form

$$(A - \lambda_i B) X_{i,j} - \lambda_{i,j} B X_i = (\lambda_i B_{,j} - A_{,j}) X_i \quad (22)$$

If an eigenvalue of Eq. (21) and its corresponding eigenvector are known while the derivatives of the eigenvalue and eigenvector are unknown then there are $n+1$ unknowns in Eq. (22); i.e., $\lambda_{i,j}$ and the n components of $X_{i,j}$ are unknown. Since the eigenvalues are assumed to be distinct then the rank of the matrix $A - \lambda_i B$ is $n-1$, the length of the eigenvector is not unique and there are only $n-1$ unique components of X_i if one component of X_i is specified. Furthermore since $A - \lambda_i B$ is a singular matrix then from Eq. (22) it is seen that the components of the derivatives of the eigenvectors are not unique.

The components of the eigenvector and its derivatives may be rendered unique by imposing the constraint on the length of the eigenvectors

$$X_i^t X_i = 1 \quad (23)$$

then

$$X_i^t X_{i,j} = 0 \quad (24)$$

Equations (22) and (24) may be written as the single matrix equation

$$\begin{bmatrix} X_i^t & \vdots & 0 \\ \cdots & \vdots & \cdots \\ A - \lambda_i B & \vdots & -BX_i \end{bmatrix} \begin{Bmatrix} X_{i,j} \\ \cdots \\ \lambda_{i,j} \end{Bmatrix} = \begin{bmatrix} 0 \\ \cdots \\ \lambda_i B_{,j} - A_{,j} \end{bmatrix} X_i \quad (25)$$

or

$$C \begin{Bmatrix} X_{i,j} \\ \cdots \\ \lambda_{i,j} \end{Bmatrix} = D X_i \quad (26)$$

The derivatives $X_{i,j}$ and $\lambda_{i,j}$ may be found by solving Eqs. (26). Matrix C may be decomposed (by means of the Choleski[6] decomposition method) into upper and lower triangular forms than a forward and backward substitution scheme may be used to evaluate $\lambda_{i,j}$ and the components of $X_{i,j}$. Since C is constant for a given eigenvalue the decomposition of C would need to be performed only one time if all $\lambda_{i,j}$ and $X_{i,j}$ (i = constant, $j = 1, 2, \ldots, m$) are to be evaluated.

Higher partial derivatives of the eigenvalues and eigenvectors may be found by differentiating Eq. (26) with respect to the independent parameter P_k, then

$$C \begin{Bmatrix} X_{i,jk} \\ \cdots \\ \lambda_{i,jk} \end{Bmatrix} = D_{,k} X_i + D X_{i,k} - C_{,k} \begin{Bmatrix} X_{i,j} \\ \cdots \\ \lambda_{i,j} \end{Bmatrix} \quad (27)$$

If $\lambda_{i,j}$, $X_{i,j}$, and $X_{i,k}$ are evaluated from Eq. (26) then the second derivatives of the eigenvalue λ_i and eigenvector X_i may be evaluated from Eq. (27) without performing a new decomposition of matrix C. The aforementioned process may be continued for any number of higher order derivatives; however, accumulating round-off error would tend to diminish the accuracy of the higher ordered derivatives.

‡ See Reference 3 for the derivation of a similar expression.

IV. Conclusions

The important feature of the methods which are presented here and in Ref. 5 for finding the derivatives of eigenvalues and eigenvectors is that these methods do not require a complete solution of the eigenvalue problem if the derivatives of only a few of the eigenvalues and eigenvectors are sought. The methods of Rudisill and Bhatia,[2] Rogers,[7] Plaut and Huseyin[8] and Fox and Kapoor[9] require all n of the eigenvalues and all n of the corresponding right-hand eigenvectors to determine the derivatives of one eigenvector of a self-adjoint-system. For non-self-adjoint systems the methods of Refs. 2 and 8 require a full set of n left-and and right-hand eigenvectors as well as all n of the eigenvalues.

The "iterative" method, which is the first of the methods to be presented here, may be used to find the first derivative of one or all of the eigenvectors along with the second derivative of their corresponding eigenvalues of a self-adjoint system in the form of Eq. (1). If the system is nonself-adjoint then only the first derivative of the eigenvector corresponding to the largest eigenvalue and the second derivative of the largest eigenvalue may be obtained, provided that the corresponding left-and right-hand eigenvectors are known. If only the first derivatives of the first m eigenvectors of a self-adjoint system are desired, then only the first m eigenvalues and their corresponding eigenvectors are needed.

The "algebraic" method which is the second method presented here for finding the derivatives of the eigenvalues and eigenvectors may be used to find all orders of the derivatives provided that they exist. If the first several orders of the derivatives of λ_j and X_j of a self-adjoint or nonself-adjoint system are needed then only the single eigenvalue λ_j and its corresponding eigenvector X_j are needed. The method of Ref. 5 in contrast requires (if the system is nonself-adjoint) in addition to λ_j and X_j the left-hand eigenvector y_j which is used to compute the derivative of the eigenvalue. The "algebraic" method which is presented here is a modification of the method of Ref. 5, and the only advantage of the method presented here is that the left-hand eigenvectors are not needed; however, this advantage is offset somewhat by the fact that the algebraic method of this Note requires the solution of $n+1$ simultaneous linear equations whereas the method of Ref. 5 requires the solution of n simultaneous equations. Garg[10] developed a method for finding the first derivatives of eigenvalues and eigenvectors of nonself-adjoint systems which was similar to the method developed here, and it did not require a complete solution to the eigenvalue problem but his method required the solution of $2(n+1)$ linear algebraic equations.

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Note on the Settling of Small Particles in a Recirculating Flow

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IN the formation of platelet aggregates or microemboli in flows containing a vortex-like separated region, the residence time of a platelet (or other small particle) within the separated part of the flow is of prime importance. One factor which could limit the residence time is the tendency of small particles to settle under the influence of gravity. In connection with this problem, we have evaluated a more general situation: the kinematics of the settling of a small particle in any recirculating flow characterized by two-dimensional closed streamlines in a vertical plane. The particle is assumed to have constant settling velocity corresponding to a small sphere moving under the influence of gravity at terminal velocity, and to have $v_s \ll u$, where v_s is the magnitude of the settling velocity and u that of the flow. For this situation, one might expect that the vertical distance displaced during a complete circulation to be simply the product of the settling velocity and the circulation time. The calculation presented here shows that this is not so, and that to a good approximation, the particle returns to its original streamline in a complete circulation, and the net vertical distance displaced is zero.

The particle motion is taken to be that of the background flow with the added constant settling velocity. The particle starts at some point on a streamline ψ_1 and moves to adjacent streamlines due to the settling motion in the negative y direction. (The x-axis is horizontal.)

Let ds be a small length along ψ_1. In traversing ds, the particle will settle a distance dy below ψ_1 given by

$$dy = -v_s(ds/u), \quad dy \ll ds \tag{1}$$

corresponding to a change in stream function

$$d\psi = (\partial\psi/\partial y)dy = u_x dy = -u_x(v_s ds/u) = -v_s(u_x/u\, ds) \tag{2}$$

or

$$d\psi = -v_s dx$$

Thus the net change in ψ during the flow with a constant v_s, is simply

$$\psi_2 - \psi_1 = -v_s(x_2 - x_1) \tag{3}$$

The result, Eq. (3), is valid to first order in v_s/u and is independent of the details of the flow. It states that when $x_2 = x_1$ then $\psi_2 = \psi_1$, and hence a particle returns to its original streamline twice in a complete circulation, and does *not* settle out of the flow. A particle "falls" inward during one part of the flow and outward during the other, the two excursions completely cancelling; independent of the details of the flow. A typical trajectory is shown in Fig. 1. (The points P and Q in the figure have the same x coordinate and indicate positions where the particle trajectory crosses the streamline ψ_1.)

The result, Eq. (3), also implies that, to first order in (v_s/u), the displacement δ normal to an initial streamline is:

$$\delta = (v_s/u)\Delta x \tag{4}$$

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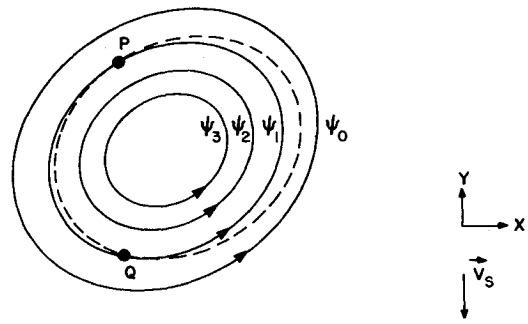

Fig. 1 Trajectory (dotted line) of a particle as it "settles" in a recirculating flow.

since $\Delta\psi = u\delta$. Thus for situations involving a vortex-like region surrounded by boundaries, (such as walls, or dividing streamlines in a separated flow) settling can affect only those particles which lie on streamlines which pass within δ of a boundary as given by Eq. (4), with Δx determined by the projection of the streamline path on the x axis. Since we are taking $v_s \ll u$, the streamlines affected represent only a small part of the flowfield. Thus, to first order in (v_s/u), settling due to gravity should be unimportant for particles in any two-dimensional recirculating flow in a vertical plane.

Localized Diamond-Shaped Buckling Patterns of Axially Compressed Cylindrical Shells

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Introduction

THE upper stability limit of cylindrical shells with classical simply supported boundary conditions was treated long ago by Flügge[1] and Pflüger.[2] This upper limit, as is well known, fails to serve as a design criterion due to the snap-through effect (Durchschlag) common in shell structures. An explanation of this snap-through characteristic and an approximate solution of this problem has been given by von Kármán et al.[3] This work which is now a classic, has been developed by many authors[2,4,5] and has also been further refined and improved in recent times, notably by Esslinger.[4] Because of the mathematical difficulty and the large algebraic labor involved, the majority of these works have dealt with simply supported boundary conditions and an over-all buckling pattern to keep the amount of calculation in reasonable limits. Indeed, the only exception known to the author is that of Ref. 5 where, unfortunately, no results were obtained because of the convergence difficulties in the large computer program.

In the following we have, therefore, the rather humble aim of obtaining only an estimation of the lower stability limit of the free edge cylinder. The method is based on a simple differential equation with Pogorelov coefficients.[6] Mechanically, the method can be interpreted as the buckling of a strut on a "bending" foundation with an attenuated eigenvector. With "bending," it is meant that the elastic constant of this fictitious foundation is due to the isometric bending in the circumferential direction.[9]

Energy Functional and the Isometric Transformation

In general, a shell surface is a two-dimensional Riemann space for which the integrability conditions are given by the

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